QFT questions

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This question set includes 9 questions on the material covered in the module, which you can use for practice as the module develops. Submit your answers to the starred questions (5, 6 and 9) for assessment after the end of the module.

1. For the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2 \phi_a \phi_a \,, \tag{1}$$

for 3 real scalar fields ϕ_a, with $a = 1, 2, 3$, verify this is invariant under the infinitesimal $SO(3)$ transformation

$$\phi_a \to \phi_a + \theta \epsilon_{abc} n_b \phi_c \,, \tag{2}$$

where θ is the rotation angle and n_b a unit vector. Compute the associated Noether current j^μ.
[Hint: There will be three of these currents.]

2. For a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2}m^2 \phi^2 \,, \tag{3}$$

show that after normal ordering the conserved 4-momentum has the operator form

$$P^\mu = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} k^\mu a^\dagger(\mathbf{k}) a(\mathbf{k}) \,. \tag{4}$$

3. Show that the integration measure

$$\frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \tag{5}$$

is Lorentz invariant, assuming that the dispersion relation satisfies $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$.
[Hint: Begin with the Lorentz invariant measure d^4k and consider how to impose the dispersion relation constraint.]

4. Show that the Feynman propagator $\Delta_F(x-y)$ is a Green function of the Klein-Gordon equation, i.e.

$$\left(\partial_\mu \partial^\mu + m^2\right) \Delta_F(x-y) = -i\delta^4(x-y) \,. \tag{6}$$

***5*.** Consider a real scalar field model with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \,. \tag{7}$$

Compute the S-matrix element for $\phi\phi \to \phi\phi$ scattering.

***6*.** In this problem you will take the non-relativistic limit of QFT to recover the familiar wavefunction and Schrödinger equation. Consider the Klein-Gordon equation for the complex scalar field with mass m and write the field as $\psi(t,\mathbf{x}) = e^{-imt}\,\tilde{\psi}(t,\mathbf{x})$ to show that

$$e^{-imt}\left[\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2\tilde{\psi}\right] = 0\,. \tag{8}$$

Now take the non-relavitistic limit $|\mathbf{p}| \ll m \Rightarrow E' \equiv E - m \ll m$. Use this to convince yourself that in this approximation $|\dot{\tilde{\psi}}| \ll m|\tilde{\psi}|$ and $|\ddot{\tilde{\psi}}| \ll m|\dot{\tilde{\psi}}|$. Hence, neglect the second time derivative term in (8) to get the non-relativistic limit of the Klein-Gordon equation:

$$i\frac{\partial\tilde{\psi}}{\partial t} = -\frac{1}{2m}\nabla^2\tilde{\psi}\,. \tag{9}$$

This looks like the Schrödinger equation for the wavefunction of a particle, but it isn't: here, $\tilde{\psi}$ is just a classical (complex) scalar field. When quantised the field should describe non-relativistic n-particle states.

Now quantise this theory: first check that equation (9) arises from the first order $\mathcal{L}$agrangian density

$$\mathcal{L} = i\tilde{\psi}^\star\dot{\tilde{\psi}} - \frac{1}{2m}\nabla\tilde{\psi}^\star\nabla\tilde{\psi} \tag{10}$$

and identify the canonically conjugate momentum $\tilde{\pi} \equiv \frac{\partial\mathcal{L}}{\partial\dot{\tilde{\psi}}}$. Thus, construct the Hamiltonian.

Repeat/adapt the quantisation procedure we followed in the relativistic case: Promote $\tilde{\psi}$ and $\tilde{\pi} = \tilde{\psi}^\star$ to operators $\hat{\psi}, \hat{\psi}^\dagger$ acting on a Hilbert space of states, and satisfying the standard commutation relations in the Shcrödinger picture. Expand the quantum field as:

$$\hat{\psi}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\hat{a}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\,, \tag{11}$$

and similarly for $\hat{\psi}^\dagger$. What commutation relations must $\hat{a}(\mathbf{p})$ and $\hat{a}^\dagger(\mathbf{p})$ satisfy? In the quantum theory, express the Hamiltonian $\hat{H}$ in terms of $\hat{a}(\mathbf{p})$ and $\hat{a}^\dagger(\mathbf{p})$. What is the commutator of $\hat{H}$ with $\hat{a}(\mathbf{p})$ and with $\hat{a}^\dagger(\mathbf{p})$? Now, define the vacuum state $|0\rangle$ and construct excited states by acting on it with $\hat{a}^\dagger(\mathbf{p})$. Use the commutator $[\hat{H}, \hat{a}^\dagger(\mathbf{p})]$ computed above to show that the one-particle states $|\mathbf{p}\rangle = \hat{a}^\dagger(\mathbf{p})|0\rangle$ satisfy

$$\hat{H}|\mathbf{p}\rangle = \frac{\mathbf{p}^2}{2m}|\mathbf{p}\rangle\,, \tag{12}$$

which is the correct dispersion relation for non-relativistic particles.

Finally, obtain the one-particle wavefunction of quantum mechanics in this field theoretic language: construct a one-particle state as a superposition of position eigenstates $|\mathbf{x}\rangle = \hat{\psi}^\dagger(\mathbf{x})|0\rangle$ as follows

$$|\varphi\rangle = \int d^3x\,\varphi(\mathbf{x})\,|\mathbf{x}\rangle\,. \tag{13}$$

Then the projection of the state onto position, $\langle x|\varphi\rangle = \varphi(\mathbf{x})$, is the wavefunction, whose time evolution $\varphi(t,\mathbf{x})$ is given by the Schrödinger equation

$$i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\nabla^2\varphi\,. \tag{14}$$

7. If γ^μ are a set of four matrices which satisfy the Clifford algebra, show that the commutator

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]\,, \tag{15}$$

satisfies the Lorentz Lie Algebra, i.e.

$$[S^{\mu\nu}, S^{\rho\sigma}] = S^{\mu\sigma}\eta^{\nu\rho} - S^{\nu\sigma}\eta^{\rho\mu} + S^{\rho\mu}\eta^{\nu\sigma} - S^{\rho\nu}\eta^{\sigma\mu}\,. \tag{16}$$

[Hint: It is usual to first compute the commutator $[S^{\mu\nu}, \gamma^\rho]$.]

8. Using the Fourier expansion of the Dirac operator $\psi(x)$ given in lectures, show that if the creation and annihilation operators satisfy the anti-commutation relations

$$\{b^r(\mathbf{k}_1), b^{s\,\dagger}(\mathbf{k}_2)\} = \{c^r(\mathbf{k}_1), c^{s\,\dagger}(\mathbf{k}_2)\} = (2\pi)^3 2E(\mathbf{k}_1)\delta^{rs}\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\,, \tag{17}$$

with all other anti-commutators vanishing, then the field and conjugate momenta satisfy

$$\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} = \{\psi^\dagger_\alpha(\mathbf{x},t), \psi^\dagger_\beta(\mathbf{y},t)\} = 0\,, \quad \{\psi_\alpha(\mathbf{x},t), \psi^\dagger_\beta(\mathbf{y},t)\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y})\,. \tag{18}$$

You may use the outer product relations $\sum_s u^s(\mathbf{k})\bar{u}^s(\mathbf{k}) = \gamma^\mu k_\mu + m$ and $\sum_s v^s(\mathbf{k})\bar{v}^s(\mathbf{k}) = \gamma^\mu k_\mu - m$.

***9*.** In relativistic Quantum Mechanics we are used to introduce 'by hand' any type of potential we wish and proceed to examine how particles react in the presence of that potential. In Quantum Field Theory we are not allowed to do this. Instead, it is the theory that predicts what sort of potential the particles experience. Since particle interactions in our relativistic field theory are described by Feynman diagrams (which are built from Feynman propagators) you may suspect that that the potential should be given by taking the non-relativistic limit of the particle propagator. This is essentially correct. In this problem you will explore how it works.

First, consider the classical theory. Can the field be interpreted as a scalar potential, whose gradient (with a minus sign) gives rise to a force? Consider a static field configuration arising from a point source in the right hand side of the equation of motion:

$$-\nabla^2\phi + m^2\phi = \delta^{(3)}(\mathbf{x}) \tag{19}$$

Take the Fourier transform of $\phi(\mathbf{x})$ to solve this equation in Fourier space. Hence, express $\phi(\mathbf{x})$ as an integral over momenta $\mathbf{k}$. To compute the integral, move to polar coordinates where $\mathbf{k}\cdot\mathbf{x} = k\,r\cos\theta$ and show that

$$\phi(\mathbf{x}) = \frac{1}{2\pi r}\mathcal{R}e\left[\int_{-\infty}^{+\infty}\frac{dk}{2\pi i}\frac{k e^{ikr}}{k^2+m^2}\right]\,. \tag{20}$$

Note this has simple poles at $k = \pm im$. Evaluate the integral using the residue theorem, closing the contour on the upper half plane, $k \to +i\infty$ so as to pick up the residue of the pole at $k = +im$. You should find:

$$\phi(\mathbf{x}) = \frac{1}{4\pi r}e^{-mr}\,, \tag{21}$$

which is the Yukawa potential. This is like Coulomb's law but with an exponential suppression, which can be understood as arising from the exchange of massive particles. But we don't know that yet as we are in classical theory.

Let's now move to the quantum theory. Can we understand the field ϕ as the potential felt by particles described by another field, say, ψ? Consider the scalar Yukawa theory we saw in the lectures (neutral scalar 'mesons' ϕ with mass m, charged scalar 'nucleons' ψ with mass M, Yukawa interaction $-g\psi^\star\psi\phi$). We have already computed the amplitude[1] for the scattering of nucleons $\psi\psi \to \psi\psi$ in Lecture 6 using Feynman rules. It arises from the exchange of mesons ϕ of mass m. We are interested in the non-relativistic limit of that amplitude. Working in the centre of momentum frame, $\mathbf{p} \equiv \mathbf{p_1} = -\mathbf{p_2}$, convince yourself that in the non-relativistic limit $|\mathbf{p}| \ll M$ the amplitude becomes:

$$i\mathcal{A} = ig^2\left[\frac{1}{(\mathbf{p}-\mathbf{q})^2+m^2} + \frac{1}{(\mathbf{p}+\mathbf{q})^2+m^2}\right] \tag{22}$$

We can now directly compare this to the scattering amplitude in Quantum Mechanics between two particles moving in a potential $U(\mathbf{r})$, where $\mathbf{r}$ is the separation between the particles. In the Born approximation we have:

$$\langle\mathbf{q}|U(\mathbf{r})|\mathbf{p}\rangle = -i\int d^3r\,U(\mathbf{r})e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}} \tag{23}$$

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[1] The amplitude $\mathcal{A}$ is related to the matrix element by $\langle \mathrm{f}|S-1|\mathrm{i}\rangle = i\mathcal{A}(2\pi)^4\delta^{(4)}(p_\mathrm{f}-p_\mathrm{i})$.

By comparing these two amplitudes you can express $U(\mathbf{r})$ as an integral over momenta. You will recover the Yukawa potential that we found in the classical theory, but now it comes with its full quantum mechanical interpretation in terms of particle exchange.

Optional (non-examinable) add-on: If you have enjoyed this calculation you can repeat it in the full Yukawa theory, where the nucleons are correctly modelled as fermions. What changes in this case? You can also try the analogous computation in Quantum Electrodynamics (QED) using the photon propagator to describe the interaction between electrons. This will recover the Coulomb force. What can you tell about the nature of the force (attractive/repulsive) in Yukawa theory and in QED?