QFT

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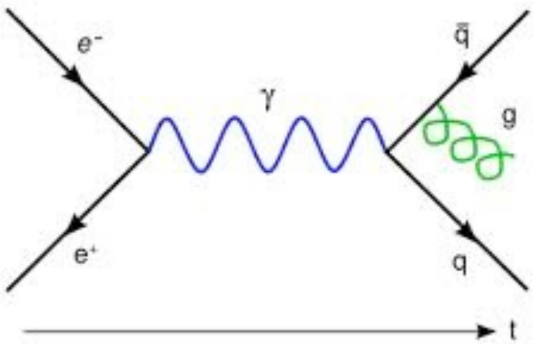
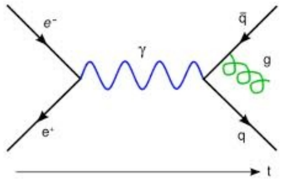
(Notes based on Dr A. Moss' lectures)



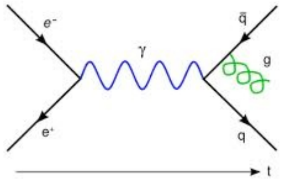
Lecture 7: Spinors

Recap: The story so far



- Relativistic QM: cannot describe changing # of particles, problems with -ve energies, -ve probabilities, causality,…
- QFT: Started with scalar field, followed canonical quantisation
- Free field corresponds to an ∞ number of harmonic oscillators: Quantum field expansion in terms of creation & annihilation ops
- Constructed complete set of eigenstates of the (free theory) Hamiltonian: n-particle states
- Creation and annihilation of particles/antiparticles needed to resolve the problems found in RQM
- Interacting Theory

Recap: Interacting Theory



- Particles briefly interact. Probability of going from $|i\rangle$ to $|f\rangle$

$$\lim_{t_\pm \to \pm\infty} \langle f|\, U(t_+, t_-)\,|i\rangle \equiv \langle f|\, S\,|i\rangle \qquad S = T \exp\left(-i\int_{-\infty}^{\infty} H_I(t')dt'\right)$$

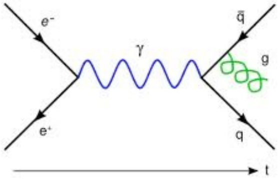
- At general order need to compute $\langle f| T\left\{H_I(x_1)\ldots H_I(x_n)\right\}|i\rangle$

- Wick's theorem: trades time ordering for normal ordering introducing Feynman propagator factors

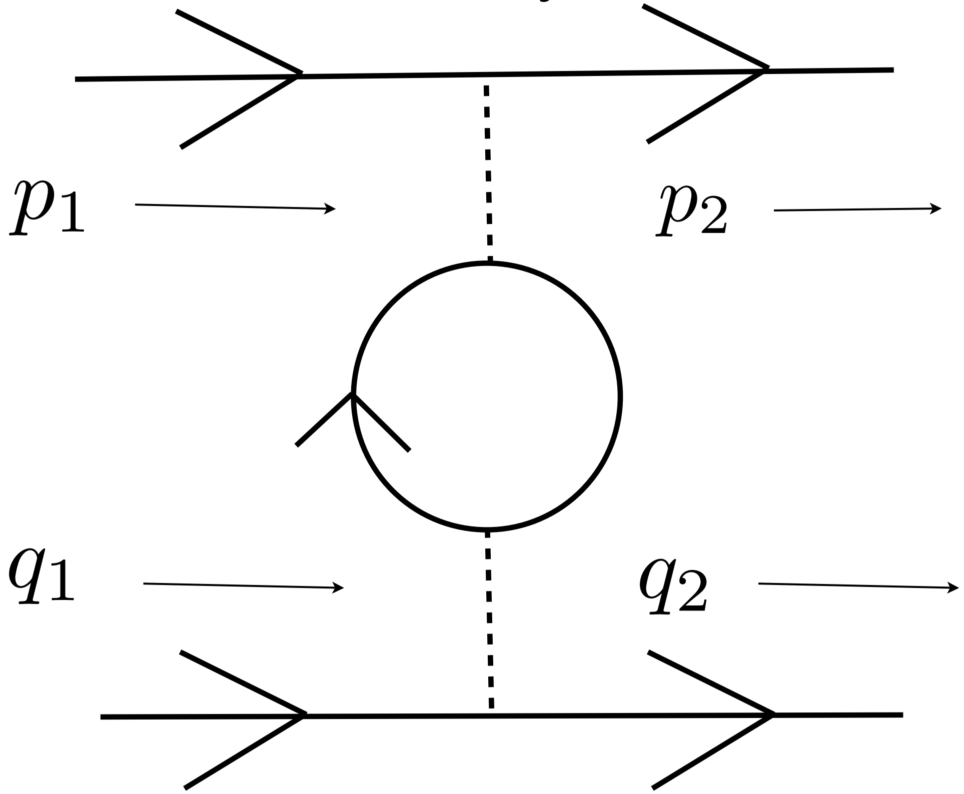
- Perturbation theory has diagrammatic representation in terms of Feynman diagrams

- Only considered *tree level* diagrams — no integrals over momenta running in loops

Higher Order Terms

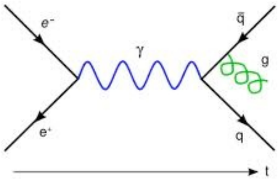


- Higher order terms can easily be considered, e.g. $O(g^4)$



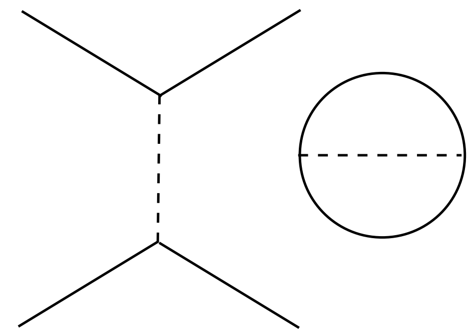
However, such diagrams involving loop momenta are often divergent → Renormalisation (not covered in this course)

Connected and Amputated Diagrams

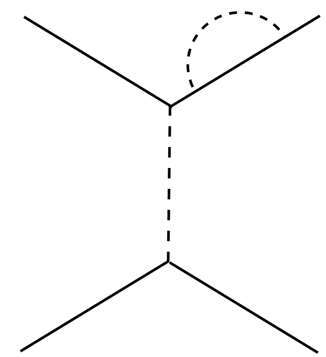


We have assumed that the initial and final states are eigenstates of the free theory Hamiltonian. This is not quite true! However, it can be dealt with as follows:

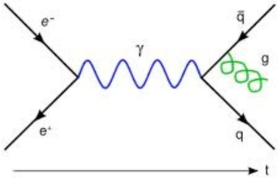
- We consider only connected diagrams, where every part is connected to external leg. Related to the fact that the true vacuum of the interacting theory is not the same as that of free theory



- Do not consider diagrams with loops on external legs. Related to the fact that one-particle states of the interacting theory are not the same as those of the free theory



Spinors

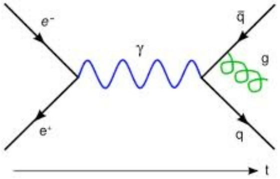


Rest of today's lecture,

Spinors:

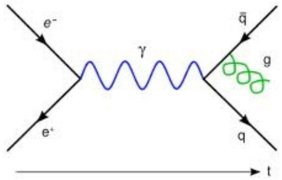
- Lorentz Group
- Spinor Representation
- Dirac Spinor
- Chiral Spinors

Lorentz Group



- So far we have only considered scalar fields
- Under a Lorentz transformation $x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu$ these transform as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. The Λ^{-1} is because we are doing an *active* transformation
- Scalar fields give rise to spin-0 particles
- To describe particles with spin (i.e. they have some intrinsic angular momentum) look at fields which have non-trivial transformations under the Lorentz group
- E.g. a vector field $A^\mu(x) \to \Lambda^\mu{}_\nu \, A^\nu(\Lambda^{-1}x)$. This gives rise to spin-1 particles

Lorentz Group



- In general a field can transform as

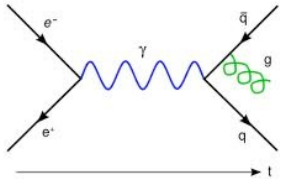
$$\phi^a(x) \to D[\Lambda]^b{}_a\, \phi^b(\Lambda^{-1}x)$$

- Here $D[\Lambda]^b{}_a$ is a matrix which depends on the Lorentz transformation (LT) we are considering. It is a *representation* of the Lorentz group
- It has the same properties as the Lorentz group, i.e.

$$D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2] \qquad D[\Lambda^{-1}] = D[\Lambda]^{-1}$$

- Want to find all possible representations such that these properties are true
- Look at infinitesimal transformations and study Lie Algebra

Lorentz Group

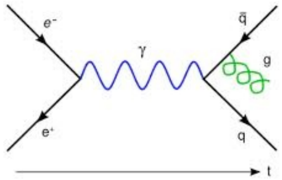


- Consider transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon w^{\mu}{}_{\nu}$
- Using definition of LT $\Lambda^{\mu}{}_{\sigma}\eta^{\sigma\tau}\Lambda^{\nu}{}_{\tau} = \eta^{\mu\nu}$
- For terms linear in ϵ then $w^{\mu\nu} + w^{\nu\mu} = 0$
- For infinitesimal LT the matrix needs to be anti-symmetric. This has 6 degrees of freedom, corresponding to the 6 transformations of the Lorentz group
- Introduce basis of 6 anti-symmetric 4x4 matrices

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$$

- ρ, σ label which matrix, μ, ν the row/column of each matrix

Lorentz Group



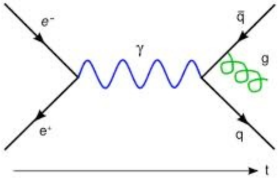
- Lower one index $(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}{}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}{}_{\nu}$
- Matrices are now no-longer antisymmetric on μ, ν
- Infinitesimal boosts:

$$(\mathcal{M}^{01})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{02})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{03})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Infinitesimal rotations:

$$(\mathcal{M}^{12})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{13})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{23})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Lorentz Group



- Can write any infinitesimal LT in terms of this basis

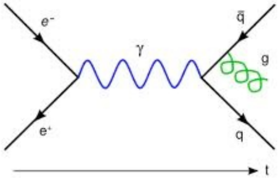
$$w^{\mu}{}_{\nu} = \frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}$$

- Here $\Omega_{\rho\sigma}$ are six real numbers specifying the LT
- Any finite LT can be written as $\Lambda^{\mu}{}_{\nu} = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}\right)$
- The six basis matrices (generators) obey the Lie algebra

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}$$

- Here the row/column index is suppressed. This equation encapsulates the properties of the Lorentz group. We are interested in other matrices which satisfy this algebra

Spinor Representation



- Interested in finding other representations of the Lorentz group
- The Clifford algebra is defined as $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1$
- γ^μ with $\mu = 0, 1, 2, 3$ are a set of 4 matrices, so

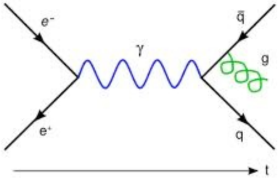
$$(\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1 \qquad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \nu \neq \mu$$

- Simplest representation is 4x4 matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

- Where σ^i are the Pauli matrices

Spinor Representation



- There is a “unique” (up to a similarity transformation) irreducible representation of the Clifford algebra. These γ^μ matrices define the chiral (or Weyl) rep

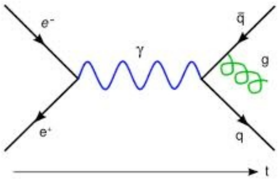
- Consider the commutator of two γ^μ

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

- Can show these form a representation of the Lorentz algebra such that

$$[S^{\rho\sigma}, S^{\tau\nu}] = \eta^{\sigma\tau}S^{\rho\nu} - \eta^{\rho\tau}S^{\sigma\nu} + \eta^{\rho\nu}S^{\sigma\tau} - \eta^{\sigma\nu}S^{\rho\tau}$$

Dirac Spinor



- Introduce a Dirac spinor, a complex valued object with 4 components which transforms as

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta\, \psi^\beta(\Lambda^{-1}x)$$

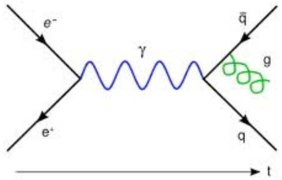
- Here $\alpha = 1, 2, 3, 4$ labels the row/column of the $S^{\mu\nu}$ matrices and

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \qquad S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

- Particular LT specified by $\Omega_{\rho\sigma}$ - these are the same for both Λ and $S[\Lambda]$

- Lets look at $S[\Lambda]$ in the chiral representation

Dirac Spinor



- For rotations $S^{ij} = \frac{1}{4}[\gamma^i, \gamma^j] = -\frac{i}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

- Writing rotation as $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$ $\quad S[\Lambda] = \begin{pmatrix} e^{i\varphi\cdot\sigma/2} & 0 \\ 0 & e^{i\varphi\cdot\sigma/2} \end{pmatrix}$

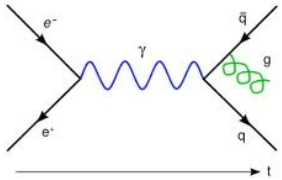
- For a rotation of $\varphi = (0, 0, 2\pi)$ $\quad S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3} \end{pmatrix} = -1$

- This means that under 2π rotations $\psi^\alpha(x) \to -\psi^\alpha(x)$ which is not what happens to a vector - different rep

- For rotations in the chiral representation $S[\Lambda]$ is unitary, i.e. $S[\Lambda]^\dagger S[\Lambda] = 1$

QFT

Dirac Spinor



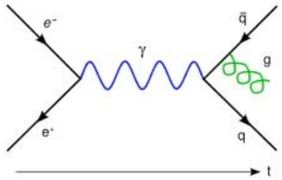
- For boosts $S^{0i} = \frac{1}{4}[\gamma^0, \gamma^i] = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

- Writing boost as $\Omega_{i0} = \chi_i$ $S[\Lambda] = \begin{pmatrix} e^{\chi\cdot\sigma/2} & 0 \\ 0 & e^{-\chi\cdot\sigma/2} \end{pmatrix}$

- For boosts in the chiral representation $S[\Lambda]$ is not unitary, i.e. $S[\Lambda]^\dagger S[\Lambda] \neq 1$

- In general there are no finite dimensional unitary representations of the Lorentz group

Chiral Spinors



- The chiral representation of the Lorentz group is reducible. It decomposes into two irreducible representations

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

- 2 component objects $u_\pm$ are called Weyl spinors

- Under rotations $u_\pm \to u_\pm e^{i\varphi\cdot\sigma/2}$

- Under boosts $u_\pm \to u_\pm e^{\pm\varphi\cdot\sigma/2}$