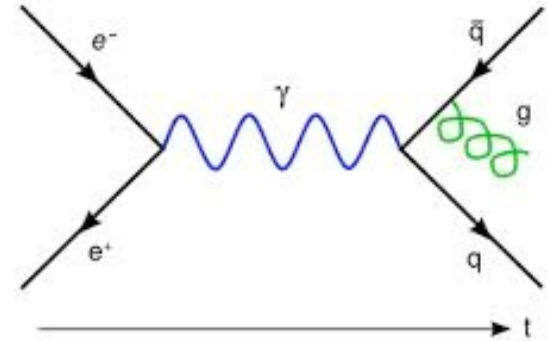


QFT

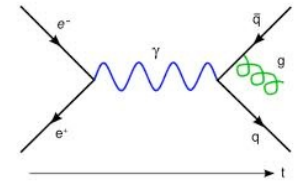
Dr Tasos Avgoustidis

(Notes based on Dr A. Moss' lectures)



Lecture 6: Interacting Fields - Wick's theorem & Feynman diagrams

Recap: Meson Decay



- $\phi \rightarrow \psi\bar{\psi}$: Initial state contains single meson of momentum p
Final state a nucleon-anti-nucleon pair of q_1 and q_2

$$|i\rangle = a^\dagger(\mathbf{p})|0\rangle \quad |f\rangle = b^\dagger(\mathbf{q}_1)c^\dagger(\mathbf{q}_2)|0\rangle$$

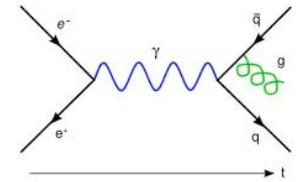
- To leading order

$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\psi(x)\phi(x)a^\dagger(\mathbf{p})|0\rangle$$

- Expand out $\phi \sim a + a^\dagger$. Only term which contributes is:

$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\psi(x)\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} a(\mathbf{k})a^\dagger(\mathbf{p})e^{-ik\cdot x}|0\rangle$$

Recap: Meson Decay



- Commute $a(\mathbf{k})$ past $a^\dagger(\mathbf{p})$ and integrate

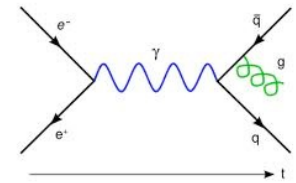
$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\psi(x)e^{-ip\cdot x}|0\rangle$$

- Similarly expand $\psi \sim b + c^\dagger$ and $\psi^\dagger \sim b^\dagger + c$

$$\langle f|S|i\rangle = -ig\langle 0|\int \frac{d^4x d^3k_1 d^3k_2}{(2\pi)^6 4E(\mathbf{k}_1)E(\mathbf{k}_2)} c(\mathbf{q}_2)b(\mathbf{q}_1)b^\dagger(\mathbf{k}_1)c^\dagger(\mathbf{k}_2)e^{i(k_1+k_2-p)\cdot x}|0\rangle$$

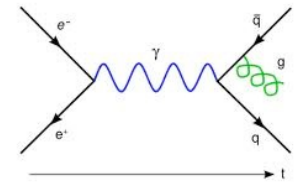
- Integrate to find $\langle f|S|i\rangle = -ig(2\pi)^4\delta^4(q_1 + q_2 - p)$
- Delta function puts constraints on decays. Boost to frame with $p = (m, 0, 0, 0)$ then $\mathbf{q}_1 = -\mathbf{q}_2$ and $m = 2\sqrt{M^2 + \mathbf{q}^2}$

Wick's Theorem



- At general order want to compute $\langle f|T \{H_I(x_1) \dots H_I(x_n)\} |i\rangle$
- Things will be a lot more convenient if we can move all annihilation operators to the right to act on $|i\rangle$
- Wick's Theorem tells us how to go from time-ordered to normal-ordered products
- For 2 fields, derived (last lecture):

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \Delta_F(x - y)$$



- Define contraction of pair in string of operators $\dots \overbrace{\phi(x_1) \dots \phi(x_2)} \dots$ to mean replace with Feynman propagator, i.e.

$$\overbrace{\phi(x)\phi(y)} = \Delta_F(x - y)$$

- For any string of operators $T[\phi(x_1) \dots \phi(x_n)] \equiv T[\phi_1 \dots \phi_n]$ Wick's Theorem states

$$T[\phi_1 \dots \phi_n] =: \phi_1 \dots \phi_n : + \text{all possible contractions} :$$

- For example 4 fields:

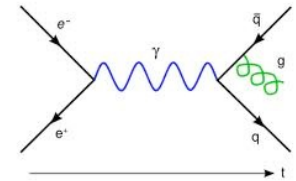
$$T[\phi_1 \phi_2 \phi_3 \phi_4] =: \phi_1 \phi_2 \phi_3 \phi_4 : +$$

$$\Delta_F(x_1 - x_2) : \phi_3 \phi_4 : + \Delta_F(x_1 - x_3) : \phi_2 \phi_4 : + \Delta_F(x_1 - x_4) : \phi_2 \phi_3 : +$$

$$\Delta_F(x_2 - x_3) : \phi_1 \phi_4 : + \Delta_F(x_2 - x_4) : \phi_1 \phi_3 : + \Delta_F(x_3 - x_4) : \phi_1 \phi_2 : +$$

$$\Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3)$$

Wick's Theorem



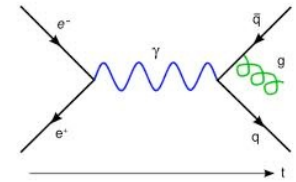
- Similar story applies for complex scalar fields. The contractions (difference between time and normal-ordered products) are

$$\overbrace{\psi(x)\psi^\dagger(y)} = \Delta_F(x-y) \quad \text{and} \quad \overbrace{\psi(x)\psi(y)} = \overbrace{\psi^\dagger(x)\psi^\dagger(y)} = 0$$

$$T\psi(x)\psi^\dagger(y) =: \psi(x)\psi^\dagger(y) : + \Delta_F(x-y)$$

$$T\psi(x)\psi(y) =: \psi(x)\psi(y) :$$

$$T\psi^\dagger(x)\psi^\dagger(y) =: \psi^\dagger(x)\psi^\dagger(y) :$$



- $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$: Initial and final states contain a nucleon-anti-nucleon pair $|i\rangle = c^\dagger(\mathbf{p}_1)b^\dagger(\mathbf{p}_2)|0\rangle$ $|f\rangle = c^\dagger(\mathbf{q}_1)b^\dagger(\mathbf{q}_2)|0\rangle$

- Contribution to S-matrix at $O(g^2)$

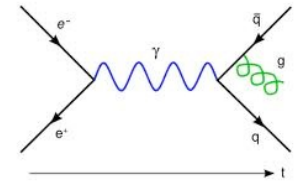
$$\frac{(-ig)^2}{2} \langle 0 | \int d^4x d^4y b(\mathbf{q}_2)c(\mathbf{q}_1) T \{ \psi^\dagger(x) \psi(x) \phi(x) \psi^\dagger(y) \psi(y) \phi(y) \} c^\dagger(\mathbf{p}_1)b^\dagger(\mathbf{p}_2) | 0 \rangle$$

- Expand time-ordered product using Wick's Theorem. Convince yourself the only term that contributes is

$$: \psi^\dagger(x) \psi(x) \psi^\dagger(y) \psi(y) : \Delta_F^\phi(x - y)$$

- Need strings of $b^\dagger c^\dagger b c$ for overlap with initial and final states. There are 4 such terms.

(can insert vacuum and use $\langle 0 | \psi(x) \psi(y) | p_1, p_2 \rangle = e^{-ip_1 \cdot x - ip_2 \cdot y} + e^{-ip_2 \cdot x - ip_1 \cdot y}$)



- Find

$$i(-ig)^2 \int \frac{d^4x d^4y d^4k}{(2\pi)^4} \frac{e^{ik \cdot (x-y)}}{k^2 - m^2 + i\epsilon} \left[e^{-i(p_1 - q_1) \cdot x} e^{-i(p_2 - q_2) \cdot y} + e^{-i(p_1 + p_2) \cdot x} e^{i(q_1 + q_2) \cdot y} \right]$$

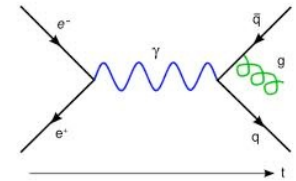
- Can now integrate to obtain amplitude

$$i(-ig)^2 \left[\frac{1}{s - m^2 + i\epsilon} + \frac{1}{t - m^2 + i\epsilon} \right] (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)$$

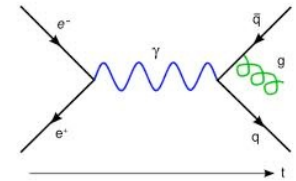
$$s = (p_1 + p_2)^2 \quad t = (p_1 - q_1)^2$$

- These are 's-channel' and t-channel' interactions - each has a simple interpretation using Feynman diagrams
- Other scattering processes such as $\psi\psi \rightarrow \psi\psi$ and $\bar{\psi}\bar{\psi} \rightarrow \bar{\psi}\bar{\psi}$ come from different strings of operators

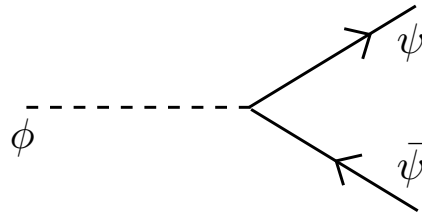
Feynman Diagrams



- Computing scattering amplitudes with Wick's Theorem is rather tedious
- Feynman diagrams provide a nice way of pictorially representing the expansion of $\langle f|S|i\rangle$
- We are interested in the terms $\langle f|S - 1|i\rangle$ (i.e. not the trivial process where no interaction occurs)
- Draw an external line for each particle in the initial and final state (choose dotted lines for mesons, solid lines for nucleons)
- Add an arrow to nucleons to denote charge (incoming arrow for ψ in initial state)



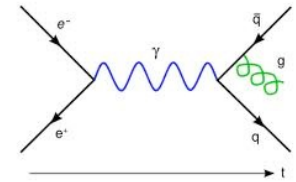
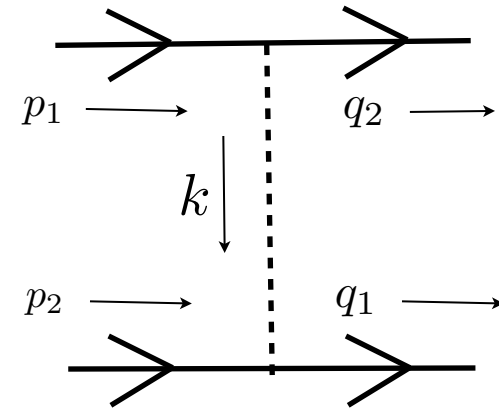
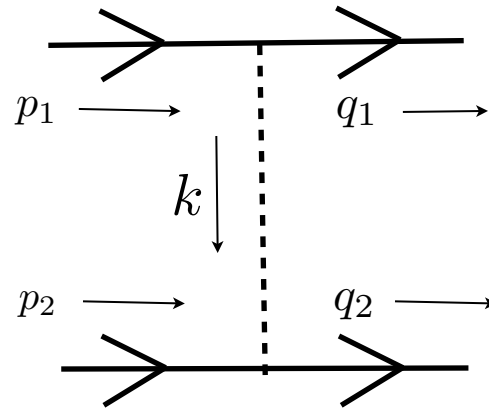
- For our scalar Yukawa theory join lines by trivalent vertices



- Add momenta k to each line
- For each vertex $(-ig)(2\pi)^4 \delta^4 \left(\sum_i k_i \right)$ where momenta are into vertex
- For each internal line integrate the propagator

$$\begin{array}{c}
 \phi \\
 \text{-----} \\
 \\
 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}
 \end{array}
 \qquad
 \begin{array}{c}
 \psi \\
 \text{-----} \\
 \longrightarrow
 \end{array}
 \qquad
 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - M^2 + i\epsilon}$$

Nucleon Scattering


 $\psi\psi \rightarrow \psi\psi$


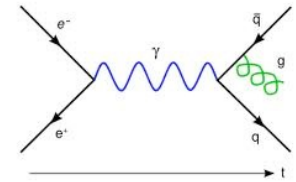
$$i(-ig)^2 \left[\frac{1}{t - m^2 + i\epsilon} + \frac{1}{u - m^2 + i\epsilon} \right] (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)$$

- t and u are Mandelstam variables:

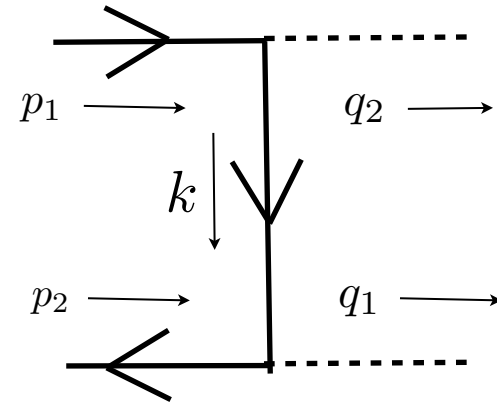
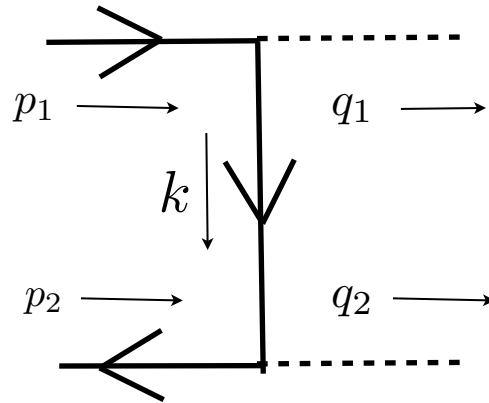
$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2 \quad u = (p_1 - q_2)^2 = (p_2 - q_1)^2$$

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2$$

Nucleon-Meson Scattering

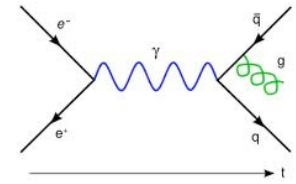


$$\psi\bar{\psi} \rightarrow \phi\phi$$

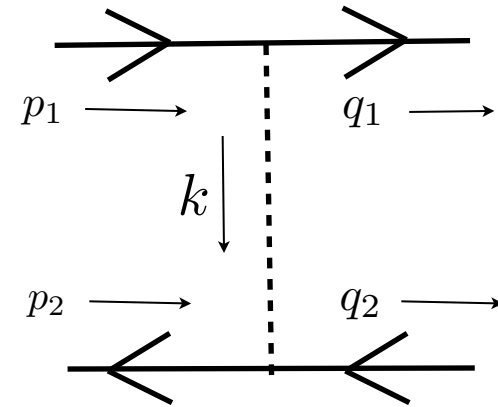
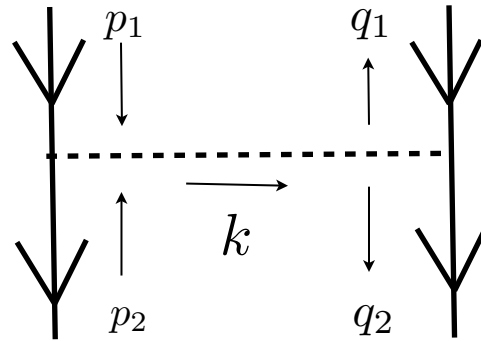


$$i(-ig)^2 \left[\frac{1}{t - M^2 + i\epsilon} + \frac{1}{u - M^2 + i\epsilon} \right] (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)$$

- Here the exchange particle is the nucleon rather than the meson
- Exchange particles do not satisfy the usual energy dispersion relation ($k^2 = m^2$ for mesons and $k^2 = M^2$ for nucleons) - we call them *virtual* particles



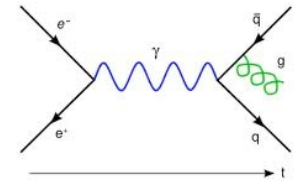
$$\psi\bar{\psi} \rightarrow \psi\bar{\psi}$$



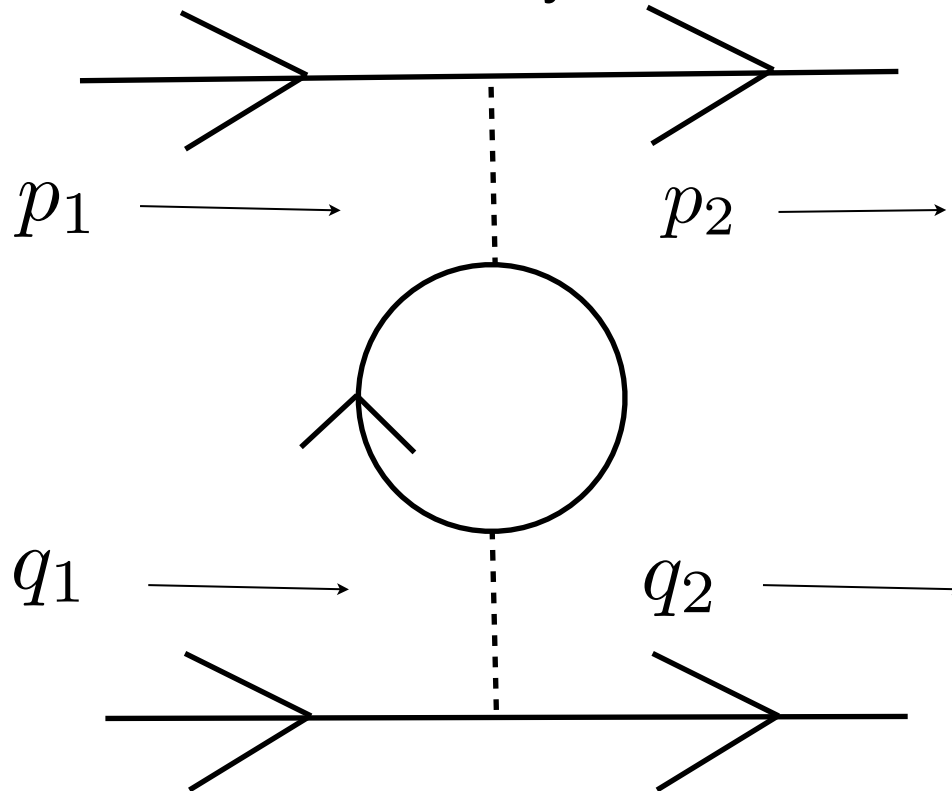
$$i(-ig)^2 \left[\frac{1}{s - m^2 + i\epsilon} + \frac{1}{t - m^2 + i\epsilon} \right] (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)$$

- If $m > 2M$ the s-channel term can diverge. However, the meson is unstable for this mass.

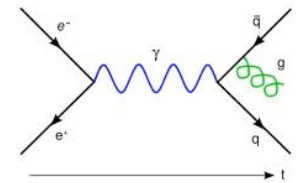
Higher Order Terms



- Higher order terms can easily be considered, e.g. $O(g^4)$



However, such diagrams involving loop momenta are often divergent \longrightarrow Renormalisation (not covered in this course)



We have assumed that the initial and final states are eigenstates of the free theory Hamiltonian. This is not quite true! However, it can be dealt with as follows:

- We consider only connected diagrams, where every part is connected to external leg. Related to the fact that the true vacuum of the interacting theory is not the same as that of free theory
- Do not consider diagrams with loops on external legs. Related to the fact that one-particle states of the interacting theory are not the same as those of the free theory

