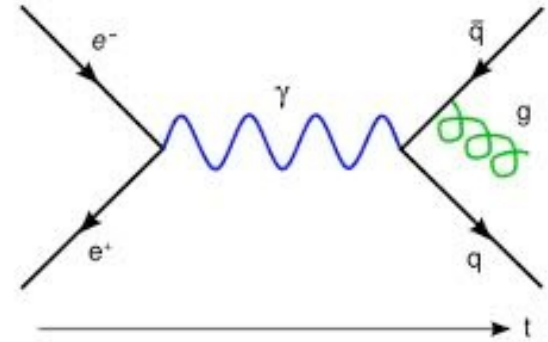


# QFT

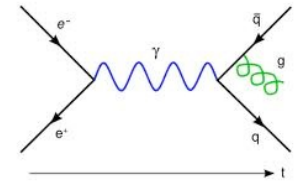
Dr Tasos Avgoustidis

(Notes based on Dr A. Moss' lectures)



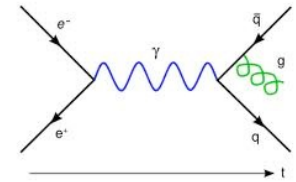
## Lecture 5: Interacting Fields

# Interacting Fields



- Will always discuss quantized fields - drop the hats off operators
- Will consider small perturbations to free theory  
Lagrangian  $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$
- Hamiltonian density of interaction  $\mathcal{H}_I = -\mathcal{L}_I$
- E.g. real scalar field  $\mathcal{L}_I = - \sum_{n \geq 3} \frac{\lambda_n}{n!} \phi^n$
- What conditions do we require on  $\lambda_n$  so the additional terms are small perturbations?

## Interacting Fields

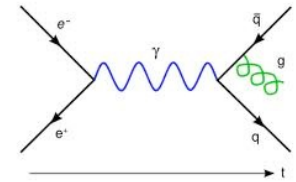


- Dimensional analysis leads to

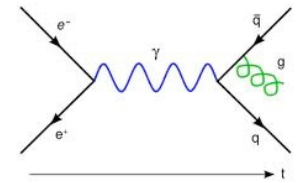
$$[\mathcal{L}] = 4 \quad [\phi] = 1 \quad [m] = 1 \quad [\lambda_n] = 4 - n$$

- $n = 3$   $[\lambda_3] = 1$ : Dimensionless parameter is  $\lambda_3/E$ . These terms are called *relevant* since they are most important at low energies. Since  $E > m$ , need  $\lambda_3 < m$
- $n = 4$   $[\lambda_4] = 0$ : Dimensionless parameter is  $\lambda_4$ . These terms are called *marginal*. Need  $\lambda_4 < 1$  for perturbative theory
- $n \geq 5$   $[\lambda_n] < 0$ : Dimensionless parameter is  $\lambda_n E^{n-4}$ . These terms are called *irrelevant* as they are suppressed at low energies

# Interacting Fields



- Irrelevant couplings can cause problems at high energies  
- lead to non-renormalizable theories
- Doesn't mean quantum theory is useless, but it is incomplete above some energy scale
- Theory still perfectly good as an Effective Field Theory at low energies - decoupling
- We will only consider theories with relevant/marginal couplings and which are weakly interacting

$\phi^4$  theory

- Lagrangian and interaction Hamiltonian:

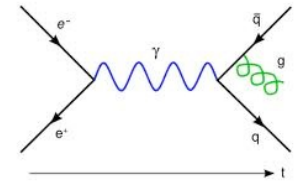
$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \quad \mathcal{H}_I = \frac{\lambda}{4!} \phi^4 \quad \lambda \ll 1$$

- Interaction term contains:

$a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})$  ,  $a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})$  , etc  
and so can create and destroy particles

- Particle number is no longer conserved  $[H, N] \neq 0$

# Scalar Yukawa Theory



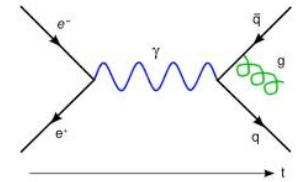
- Lagrangian and interaction Hamiltonian:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \partial_\mu \psi^* \partial^\mu \psi - \frac{1}{2} m^2 \phi^2 - M^2 \psi^* \psi - g \psi^* \psi \phi$$

$$\mathcal{H}_I = g \psi^* \psi \phi \quad g \ll M, m$$

- Individual particle numbers not conserved
- Symmetry ensuring number of  $\psi$  particles minus anti-particles (denoted  $\bar{\psi}$ ) is conserved

# Dyson's Formula

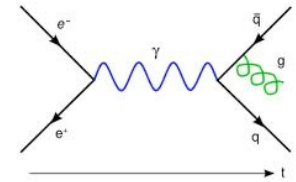


- Recall in interaction picture  $i \frac{\partial}{\partial t} |\psi(t)\rangle_I = H_I(t) |\psi(t)\rangle_I$
- Write solution in terms of unitary time evolution operator such that  $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$  and  $U(t, t) = 1$

$$|\psi(t)\rangle_I = U(t, t_0) |\psi(t_0)\rangle_I$$

- Requires that  $i \frac{dU}{dt} = H_I(t)U$
- If  $H_I(t)$  were a function  $U(t, t_0) = \exp \left( -i \int_{t_0}^t H_I(t') dt' \right)$
- Since it is an operator there are ordering issues (apparent when expanding the exponential in powers of  $H_I(t)$ )

# Dyson's Formula



- To solve integrate time evolution equation

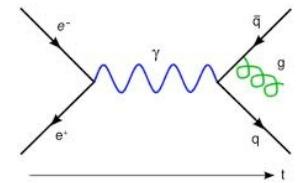
$$U(t, t_0) = 1 - i \int_{t_0}^t H_I(t_1) U(t_1, t_0) dt_1$$

- Still depends on evolution operator. Substitute new expression into integrand (  $t_2 < t_1 < t$  )

$$U(t, t_0) = 1 - i \int_{t_0}^t H_I(t_1) dt_1 - \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) \dots$$

- Can be written as time-ordered exponential

$$U(t, t_0) = T \exp \left( -i \int_{t_0}^t H_I(t') dt' \right) \quad T \hat{\phi}(x) \hat{\phi}(y) = \begin{cases} \hat{\phi}(x) \hat{\phi}(y), & \text{if } x^0 > y^0 \\ \hat{\phi}(y) \hat{\phi}(x), & \text{if } y^0 > x^0 \end{cases}$$

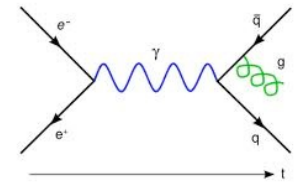


*Assume initial & final states are eigenstates of free theory*

- Initial state  $|i\rangle$  at  $t = -\infty$  and final state  $|f\rangle$  at  $t = +\infty$  are eigenstates of free Hamiltonian  $H_0$
- At  $t = \pm\infty$  particles are far separated and don't feel the effects of each other
- Initial and final states are eigenstates of number operator with  $[H_0, N] = 0$ , but crucially  $[H_I, N] \neq 0$
- Particles briefly interact. Probability of going from  $|i\rangle$  to  $|f\rangle$

$$\lim_{t_{\pm} \rightarrow \pm\infty} \langle f | U(t_+, t_-) | i \rangle \equiv \langle f | S | i \rangle \quad S = T \exp \left( -i \int_{-\infty}^{\infty} H_I(t') dt' \right)$$

# Scalar Yukawa Theory

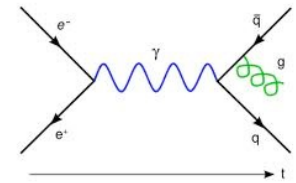


- Lets look at our scalar Yukawa theory

$$H_I = g \int d^3x \psi^\dagger(x) \psi(x) \phi(x)$$

- Interaction Hamiltonian contains
  - $\phi \sim a + a^\dagger$  which can create/destroy  $\phi$  particles (call these mesons)
  - $\psi \sim b + c^\dagger$  which can create  $\bar{\psi}$  and destroy  $\psi$  (call these nucleons)
  - $\psi^\dagger \sim b^\dagger + c$  which can create  $\psi$  and destroy  $\bar{\psi}$

## Meson Decay



- $\phi \rightarrow \psi\bar{\psi}$ : Initial state contains single meson of momentum  $p$   
Final state a nucleon-anti-nucleon pair of  $q_1$  and  $q_2$

$$|i\rangle = a^\dagger(\mathbf{p})|0\rangle \quad |f\rangle = b^\dagger(\mathbf{q}_1)c^\dagger(\mathbf{q}_2)|0\rangle$$

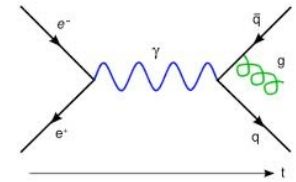
- To leading order

$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\psi(x)\phi(x)a^\dagger(\mathbf{p})|0\rangle$$

- Expand out  $\phi \sim a + a^\dagger$ . Only term which contributes is:

$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\psi(x)\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} a(\mathbf{k})a^\dagger(\mathbf{p})e^{-ik\cdot x}|0\rangle$$

# Meson Decay



- Commute  $a(\mathbf{k})$  past  $a^\dagger(\mathbf{p})$  and integrate

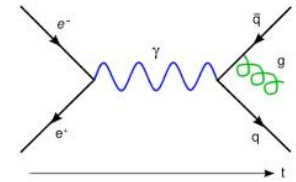
$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\psi(x)e^{-ip\cdot x}|0\rangle$$

- Similarly expand  $\psi \sim b + c^\dagger$  and  $\psi^\dagger \sim b^\dagger + c$

$$\langle f|S|i\rangle = -ig\langle 0|\int \frac{d^4x d^3k_1 d^3k_2}{(2\pi)^6 4E(\mathbf{k}_1)E(\mathbf{k}_2)} c(\mathbf{q}_2)b(\mathbf{q}_1)b^\dagger(\mathbf{k}_1)c^\dagger(\mathbf{k}_2)e^{i(k_1+k_2-p)\cdot x}|0\rangle$$

- Integrate to find  $\langle f|S|i\rangle = -ig(2\pi)^4\delta^4(q_1 + q_2 - p)$
- Delta function puts constraints on decays. Boost to frame with  $p = (m, 0, 0, 0)$  then  $\mathbf{q}_1 = -\mathbf{q}_2$  and  $m = 2\sqrt{M^2 + \mathbf{q}^2}$

## Wick's Theorem

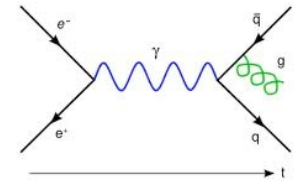


- At general order want to compute  $\langle f|T \{H_I(x_1) \dots H_I(x_n)\} |i\rangle$
- Things will be a lot more convenient if we can move all annihilation operators to the right to act on  $|i\rangle$
- Wick's Theorem tells us how to go from time-ordered to normal-ordered products
- Consider scalar field  $\phi(x) = \phi^+(x) + \phi^-(x)$

with:

$$\phi^+(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} a(\mathbf{k}) e^{-ik \cdot x}$$

$$\phi^-(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} a^\dagger(\mathbf{k}) e^{ik \cdot x}$$



- When  $x^0 > y^0$

$$T\phi(x)\phi(y) = \phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^-(y) + [\phi^+(x), \phi^-(y)]$$

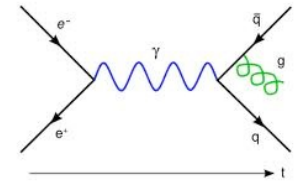
- Commutator is equal to the propagator  $D(x - y)$

$$\text{Recall } D(x - y) = \int \frac{d^3 k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik \cdot (x - y)}$$

- Time ordered product is therefore

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D(x - y)$$

## Wick's Theorem



- Case when  $y^0 > x^0$

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D(y - x)$$

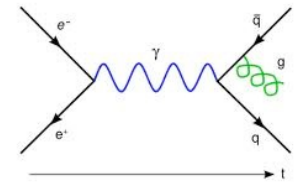
$$\text{with } D(y - x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik \cdot (y-x)}$$

- Putting together for 2 fields:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \Delta_F(x - y)$$

$$\text{Recall } \Delta_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik \cdot (x-y)}$$

## Wick's Theorem



- Define contraction of pair in string of operators  $\dots \overbrace{\phi(x_1) \dots \phi(x_2)} \dots$  to mean replace with Feynman propagator, i.e.

$$\overbrace{\phi(x)\phi(y)} = \Delta_F(x - y)$$

- For any string of operators  $T[\phi(x_1) \dots \phi(x_n)] \equiv T[\phi_1 \dots \phi_n]$  Wick's Theorem states

$$T[\phi_1 \dots \phi_n] =: \phi_1 \dots \phi_n : + \text{all possible contractions} :$$

- For example 4 fields:

$$T[\phi_1 \phi_2 \phi_3 \phi_4] =: \phi_1 \phi_2 \phi_3 \phi_4 : +$$

$$\Delta_F(x_1 - x_2) : \phi_3 \phi_4 : + \Delta_F(x_1 - x_3) : \phi_2 \phi_4 : + \Delta_F(x_1 - x_4) : \phi_2 \phi_3 : +$$

$$\Delta_F(x_2 - x_3) : \phi_1 \phi_4 : + \Delta_F(x_2 - x_4) : \phi_1 \phi_3 : + \Delta_F(x_3 - x_4) : \phi_1 \phi_2 : +$$

$$\Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3)$$