QFT

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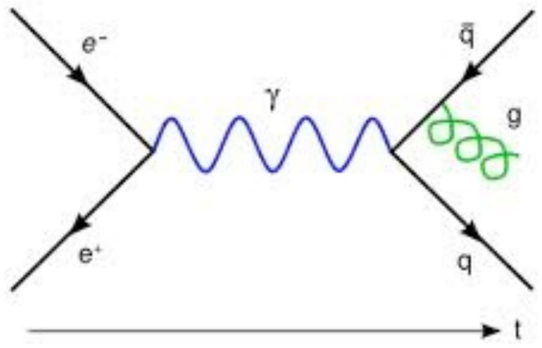
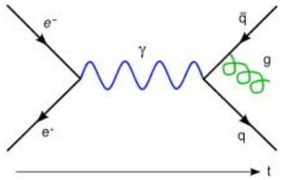
(Notes based on Dr A. Moss' lectures)



Lecture 2: Preliminaries (Quantum)

Noether's Theorem



- Symmetry: transformation $\delta\phi = X(\phi)$ such that

$$\delta\mathcal{L} = \partial_\mu F^\mu(\phi) \text{ (total derivative)}$$

- Change in Lagrangian is $$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)$$

- Euler-Lagrange equations give $$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)$$
(for any variation, including $\delta\phi = X(\phi)$)

- Thus, there is conserved current:

$$\partial_\mu j^\mu = 0 \qquad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi) - F^\mu(\phi)$$

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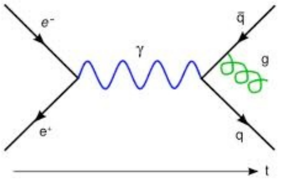
Noether's Theorem

for translational invariance



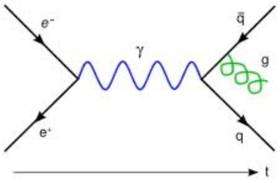
- Consider infinitesimal translation $x^\mu \to x^\mu + \epsilon^\mu$
- Change in Lagrangian is $\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)$
- Euler-Lagrange equations give $\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)$
- Under translation $\phi(x) \to \phi(x) - \epsilon^\mu\partial_\mu\phi(x)$

$$\mathcal{L}(x) \to \mathcal{L}(x) - \epsilon^\mu\partial_\mu\mathcal{L}(x)$$

NB Lagrangian has no explicit coordinate dependence

Noether's Theorem

for translational invariance



- For invariance of action for general ϵ^μ find 4 conserved currents

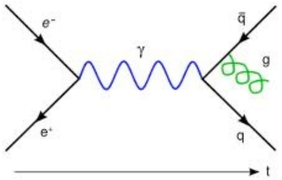
$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \equiv T^\mu{}_\nu$$

- $T^\mu{}_\nu$ is the energy-momentum tensor which satisfies

$$\partial_\mu T^\mu{}_\nu = 0$$

- Translation symmetry gives rise to conservation of energy-momentum
- Other symmetries give other conserved currents - e.g. Lorentz transformation and angular momentum

Energy-Momentum Tensor



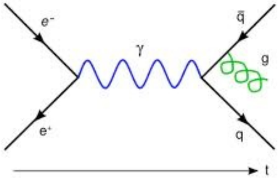
- 4 conserved quantities - energy and total momentum of field

$$E = \int d^3x \, T^{00} \qquad P^i = \int d^3x \, T^{0i}$$

- Identify T^{00} as the Hamiltonian density
- For scalar field theory with $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$
- Energy-momentum tensor $T^{\mu\nu} = \partial^\mu\phi \, \partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$
- Find conserved energy and momentum

$$E = \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \qquad P^i = \int d^3x \, \dot{\phi} \, \partial^i\phi$$

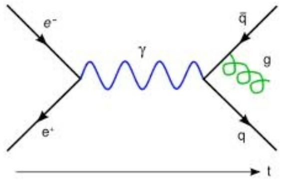
Canonical Quantization



- In quantum mechanics canonical quantization takes Hamiltonian formalism of classical mechanics to quantum theory
  - Dynamical variables such as position x_i and momentum p_i are promoted to operators
  - Poisson bracket structure of classical mechanics morphs into commutation relations

- Recall Hamilton's equations $\frac{\partial H}{\partial x_i} = -\dot{p}^i \, , \quad \frac{\partial H}{\partial p^i} = \dot{x}_i$

Canonical Quantization

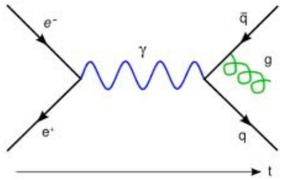


- For observable $\dot{\mathcal{O}}(x,p) = \frac{\partial \mathcal{O}}{\partial x_i}\frac{\partial H}{\partial p^i} - \frac{\partial \mathcal{O}}{\partial p^i}\frac{\partial H}{\partial x_i} = \{\mathcal{O}, H\}$
- Poisson bracket $\{x_i, x_j\} = \{p^i, p^j\} = 0 \quad \{x_i, p^j\} = \delta_i^j$
- Classical to quantum $\{\,,\}_{\text{classical}} \to -i\,[\,,]_{\text{quantum}}$
- Commutation relations

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}^i, \hat{p}^j] = 0 \qquad [\hat{x}_i, \hat{p}^j] = i\delta_i^j$$

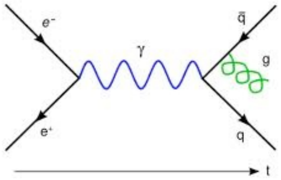
- In field theory will do the same for field $\phi(x)$ and momentum conjugate $\pi(x)$
- Will first do this in the *Schrödinger* picture. In *Heisenberg* picture these will be *equal time* commutation relations

State Vectors



- Physical states are encoded in state vector $|\psi\rangle$ in Hilbert space $\mathcal{H}$
- Eigenstates of an operator defined by $\hat{A}|\psi\rangle = a|\psi\rangle$
- Measurable quantities given by expectation value of Hermitian operators $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$
- Hermiticity ensures expectation values are real
- Probability to go from state 1 to state 2 $|\langle\psi_1|\psi_2\rangle|^2$
- Eigenstates form a complete orthonormal basis - can expand arbitrary state vector in set of eigenstates

Schrödinger Picture



- State vectors are functions of time, while operators are time independent

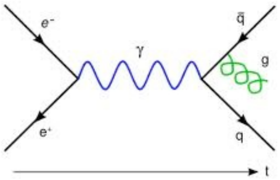
- Time evolution described by Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

- Time dependent state vector

$$|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)}|\psi(t_0)\rangle$$

Heisenberg Picture



- State vectors regarded as constant and operators carry time dependence

- State vector defined as

$$|\psi(t)\rangle_S = e^{-i\hat{H}(t-t_0)}|\psi(t_0)\rangle_H$$

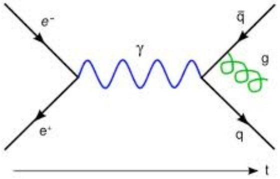
- Transformation should leave expectation values invariant
- Define Heisenberg operator

$$\hat{O}_H(t) = e^{i\hat{H}(t-t_0)}\hat{O}_S e^{-i\hat{H}(t-t_0)}$$

- Heisenberg equation of motion for operators

$$i\frac{d\hat{O}_H(t)}{dt} = \left[\hat{O}_H, \hat{H}\right]$$

Interaction Picture



- Split up Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$
- Useful when we have small perturbations to a well-understood Hamiltonian (later $\hat{H}_0$ will be Hamiltonian of free field theory)
- Time dependence of operators governed by $\hat{H}_0$ and time dependence of states by $\hat{H}_{\text{int}}$
- Define

$$|\psi(t)\rangle_I = e^{i\hat{H}_0(t-t_0)}|\psi(t)\rangle_S$$

$$\hat{O}_I(t) = e^{i\hat{H}_0(t-t_0)}\hat{O}_S e^{-i\hat{H}_0(t-t_0)}$$

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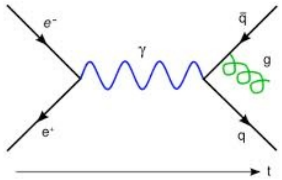
Interaction Picture



- Interaction Hamiltonian in interaction picture

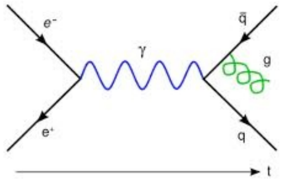
$$\hat{H}_I(t) \equiv (\hat{H}_{\rm int})_I(t) = e^{i\hat{H}_0(t-t_0)}(\hat{H}_{\rm int})_S e^{-i\hat{H}_0(t-t_0)}$$

- Schrödinger equation for states

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_S = \hat{H}_S|\psi(t)\rangle_S \quad \rightarrow \quad i\frac{\partial}{\partial t}|\psi(t)\rangle_I = \hat{H}_I(t)|\psi(t)\rangle_I$$

- Later we will solve this equation but will have to deal with ordering issues

Harmonic Oscillator



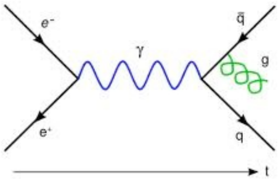
- General solution to Klein-Gordon equation is linear superposition of HOs, as we will see. Recall Quantum HO:
- Hamiltonian given by $\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right)$
- Introduce new operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\,\hat{x} + i\sqrt{\frac{1}{m\omega}}\,\hat{p}\right) \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\,\hat{x} - i\sqrt{\frac{1}{m\omega}}\,\hat{p}\right)$$

- Find commutation relations

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 \qquad \left[\hat{H}, \hat{a}^\dagger\right] = \omega\hat{a}^\dagger \qquad \left[\hat{H}, \hat{a}\right] = -\omega\hat{a}$$

Harmonic Oscillator



- Rewrite Hamiltonian as $\hat{H} = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$
- Can construct complete basis of energy eigenstates $|n\rangle$

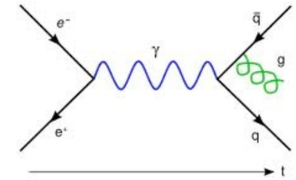
$$\hat{H}|n\rangle = E_n|n\rangle$$

- Using commutation relations

$$\hat{H}\hat{a}^\dagger|n\rangle = (E_n + \omega)\,\hat{a}^\dagger|n\rangle \qquad \hat{H}\hat{a}|n\rangle = (E_n - \omega)\,\hat{a}|n\rangle$$

- Creation and annihilation operators, raising/lowering energy
- Define ground state by $\hat{a}|0\rangle = 0$
- Zero-point energy $\hat{H}|0\rangle = \frac{\omega}{2}|0\rangle$
- Excited states: repeated application of $\hat{a}^\dagger$ on ground state

Klein-Gordon Equation



- Recall Lagrangian $$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$
- Euler-Lagrange equation then gives Klein-Gordon equation $$\partial_\mu\partial^\mu\phi + m^2\phi = \left(\Box + m^2\right)\phi = 0$$
- Expand in Fourier modes $$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\,\phi(\mathbf{k},t)$$ and notice $\phi(\mathbf{k},t)$ satisfies: $$\left[\frac{\partial^2}{\partial t^2} + \left(\mathbf{k}^2 + m^2\right)\right]\phi(\mathbf{k},t) = 0$$
- Infinite number of HO's with frequency $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$ (labelled by $\mathbf{k}$)

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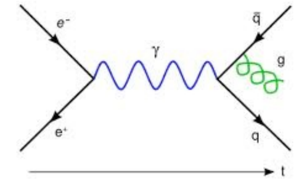
Canonical Quantization



- Analogous to quantum mechanics promote canonical variables to be operators acting on states

$$\phi(\mathbf{x}) \rightarrow \hat{\phi}(\mathbf{x}) \qquad \pi(\mathbf{x}) \rightarrow \hat{\pi}(\mathbf{x})$$

- Impose commutation relations (from Poisson brackets)

$$\left[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})\right] = \left[\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = 0$$

$$\left[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = i\delta^3(\mathbf{x} - \mathbf{y})$$

- 3 dimensional δ-function as we are using fields
- Note we are in the *Schrödinger picture*: operators depend only on space, all time dependence is in the states