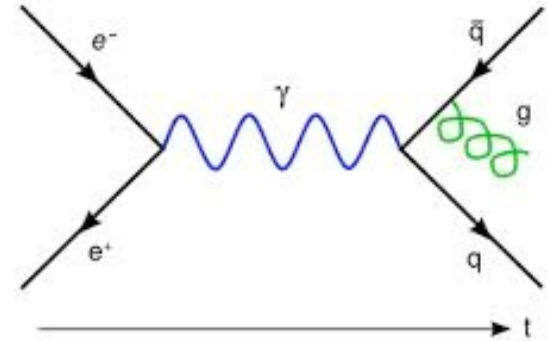


QFT

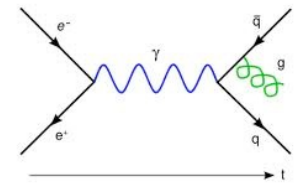
Dr Tasos Avgoustidis

(Notes based on Dr A. Moss' lectures)



Lecture 10: Interacting Dirac Field - Feynman Diagrams

Nucleon-Anti-Nucleon Scattering



- $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$: Initial and final state contains a nucleon-anti-nucleon pair $|i\rangle = b^{s_1 \dagger}(\mathbf{p}_1)c^{s_2 \dagger}(\mathbf{p}_2)|0\rangle$, $|f\rangle = b^{r_1 \dagger}(\mathbf{q}_1)c^{r_2 \dagger}(\mathbf{q}_2)|0\rangle$
- Contribution to S-matrix at $O(g^2)$

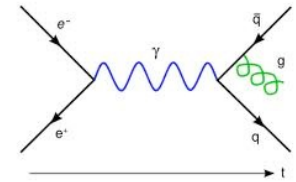
$$\frac{(-ig)^2}{2} \langle 0 | \int d^4x d^4y c^{r_2}(\mathbf{q}_2) b^{r_1}(\mathbf{q}_1) T \{ \bar{\psi}(x) \psi(x) \phi(x) \bar{\psi}(y) \psi(y) \phi(y) \} b^{s_1 \dagger}(\mathbf{p}_1) c^{s_2 \dagger}(\mathbf{p}_2) | 0 \rangle$$

- As in bosonic case only term which contributes in time-ordered product is

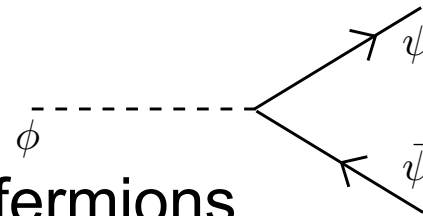
$$: \bar{\psi}(x) \psi(x) \bar{\psi}(y) \psi(y) : \Delta_F^\phi(x - y)$$

- Have to be careful with spinor indices - calculation is quite tedious (try it!)

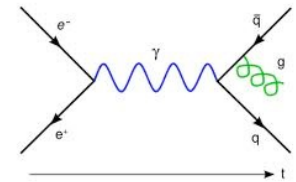
Feynman Rules



- Draw an external line for each particle in the initial and final state (as before will choose dotted lines for mesons, solid lines for nucleons)
- Add an arrow to nucleons to denote charge (incoming arrow for ψ in initial state)
- Join lines by trivalent vertices
- Associate spinors with external fermions
 - For incoming nucleon $u^s(\mathbf{k})$
 - For outgoing nucleon $\bar{u}^s(\mathbf{k})$
 - For incoming anti-nucleon $\bar{v}^s(\mathbf{k})$
 - For outgoing anti-nucleon $v^s(\mathbf{k})$



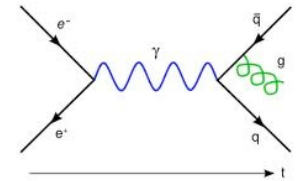
Feynman Rules



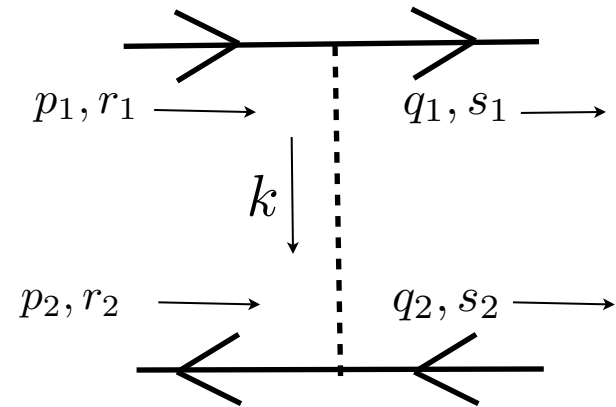
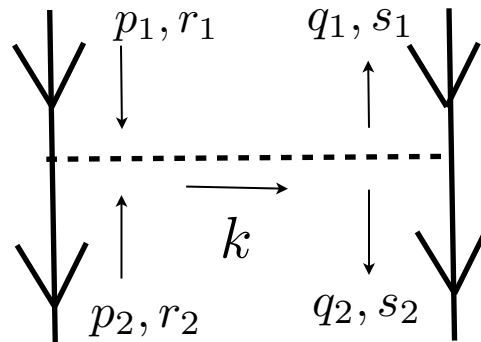
- For each vertex $(-ig)(2\pi)^4 \delta^4 \left(\sum_i k_i \right)$ where momenta are into vertex
- For each internal line integrate the propagator

$$\begin{array}{ccc}
 \phi & & \psi \\
 \text{-----} & & \text{-----} \\
 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} & & \int \frac{d^4 k}{(2\pi)^4} \frac{i(\gamma^\mu k_\mu + M)}{k^2 - M^2 + i\epsilon}
 \end{array}$$

- Nucleon propagator is now a 4x4 matrix
- Spinor indices are contracted at each vertex
- Add minus signs for statistics



$$\psi\bar{\psi} \rightarrow \psi\bar{\psi}$$



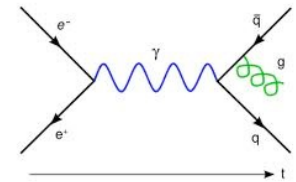
- Define amplitude by $\langle f|S - 1|i\rangle = i\mathcal{A}(2\pi)^4\delta^4(k_F - k_I)$

$$\mathcal{A} = (-ig)^2 \left[\frac{[\bar{v}^{r_2}(\mathbf{p}_2) \cdot u^{r_1}(\mathbf{p}_1)][\bar{u}^{s_2}(\mathbf{q}_2) \cdot v^{s_1}(\mathbf{q}_1)]}{s - m^2 + i\epsilon} - \frac{[\bar{u}^{s_1}(\mathbf{q}_1) \cdot u^{r_1}(\mathbf{p}_1)][\bar{v}^{r_2}(\mathbf{p}_2) \cdot v^{s_2}(\mathbf{q}_2)]}{t - m^2 + i\epsilon} \right]$$

$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2 \quad u = (p_1 - q_2)^2 = (p_2 - q_1)^2$$

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2$$

- If $m > 2M$ the s-channel term can again diverge. However, the meson is unstable for this mass



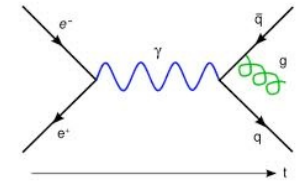
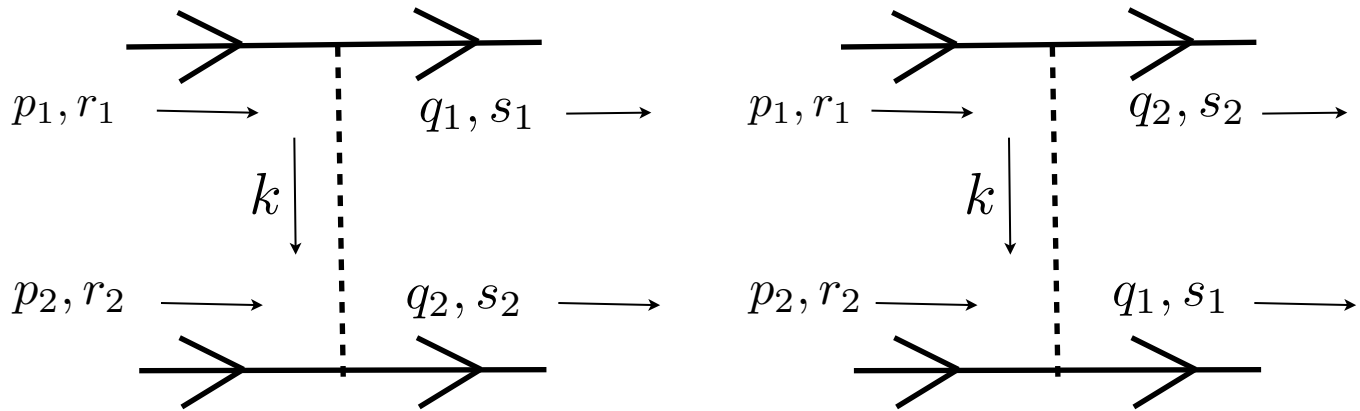
- The minus signs can be a little tricky to get right
- Safest thing to do is to go back to the calculation of the S-matrix element using Wick's theorem
- For the s-channel of nucleon-anti-nucleon scattering this is given (very) schematically by

$$\begin{aligned} \langle f | : \bar{\psi} \psi \bar{\psi} \psi : | i \rangle &= \langle 0 | cb : \bar{v} c u b \bar{u} b^\dagger v c^\dagger : b^\dagger c^\dagger | 0 \rangle \\ &= + \langle 0 | c b b^\dagger c^\dagger [\bar{v} u] [\bar{u} v] c b b^\dagger c^\dagger | 0 \rangle = + [\bar{v} u] [\bar{u} v] \end{aligned}$$

- The t-channel term is

$$\begin{aligned} \langle f | : \bar{\psi} \psi \bar{\psi} \psi : | i \rangle &= \langle 0 | cb : \bar{v} c v c^\dagger \bar{u} b^\dagger u b : b^\dagger c^\dagger | 0 \rangle \\ &= + \langle 0 | c b c^\dagger b^\dagger [\bar{v} v] [\bar{u} u] c b b^\dagger c^\dagger | 0 \rangle = - [\bar{v} v] [\bar{u} u] \end{aligned}$$

Nucleon Scattering


 $\psi\psi \rightarrow \psi\psi$


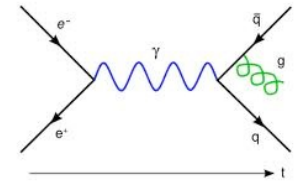
$$\mathcal{A} = (-ig)^2 \left[\frac{[\bar{u}^{s_1}(\mathbf{q}_1) \cdot u^{r_1}(\mathbf{p}_1)][\bar{u}^{s_2}(\mathbf{q}_2) \cdot u^{r_2}(\mathbf{p}_2)]}{t - m^2 + i\epsilon} - \frac{[\bar{u}^{s_2}(\mathbf{q}_2) \cdot u^{r_1}(\mathbf{p}_1)][\bar{u}^{s_1}(\mathbf{q}_1) \cdot u^{r_2}(\mathbf{p}_2)]}{u - m^2 + i\epsilon} \right]$$

- Notice relative minus sign

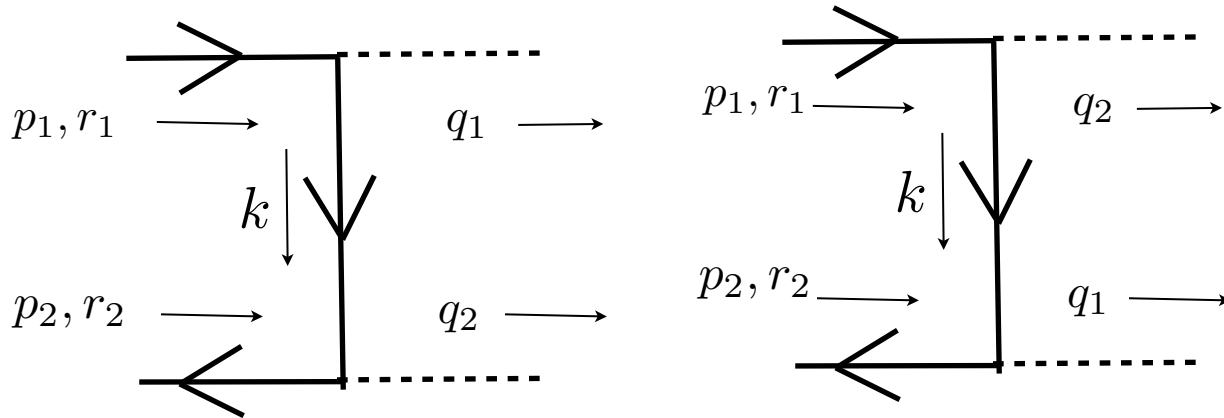
(Also notice relative minus sign compared to t-channel diagram for $\psi\bar{\psi} \rightarrow \psi\bar{\psi}$)

(cf. scalar Yukawa theory result in lecture 6)

Nucleon-Meson Scattering



$$\psi\bar{\psi} \rightarrow \phi\phi$$

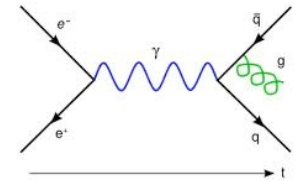


$$\mathcal{A} = (-ig)^2 \left[\frac{\bar{v}^{r_2}(\mathbf{p}_2) [\gamma \cdot (p_1 - q_1) + M] u^{r_1}(\mathbf{p}_1)}{t - M^2 + i\epsilon} + \frac{\bar{v}^{r_2}(\mathbf{p}_2) [\gamma \cdot (p_1 - q_2) + M] u^{r_1}(\mathbf{p}_1)}{u - M^2 + i\epsilon} \right]$$

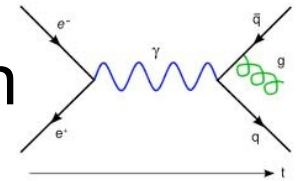
- Here exchange particle is nucleon rather than meson
- Final states are mesons: no relative minus sign

(cf. scalar Yukawa theory result in lecture 6)

The photon field and QED

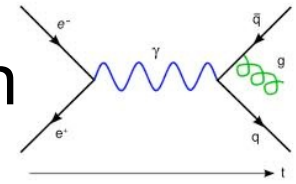


- In this course we have not quantised the vector field A_μ whose excitations are photons (quantisation proceeds in a similar way, but there are new subtleties)
- Its coupling to matter is determined by symmetry: demand that the global U(1) symmetry of our fermion Lagrangian survives for spacetime-dependent parameter
 - Find that this can be maintained if we change our partial derivative to a covariant derivative, linear in a vector field that is subject to gauge transformations
- Studied in the QED course



- Canonical quantisation of vector field is subtle due to gauge invariance (easier to perform functional quantisation)
- Consider Maxwell equations $\partial_\mu F^{\mu\nu} = 0$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \partial_{[\lambda} F_{\mu\nu]} = 0$$
 and note:
 - No time derivative in A_0
 - Gauge invariance $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x)$
- Expect 4-2 degrees of freedom (2 polarisations for photons)
- However, it's quite subtle to obtain in quantum theory:
 - In Coulomb gauge, subtlety is in modified Poisson brackets due to constraint. D.o.f. manifest but no Lorentz invariance.
 - Let's have a quick look at quantisation in Lorenz gauge. Subtlety is in how to impose the gauge to identify d.o.f.



- Work in Lorentz gauge $\partial_\mu A^\mu = 0$

where e.o.m. reads $\partial_\mu \partial^\mu A^\nu = 0$

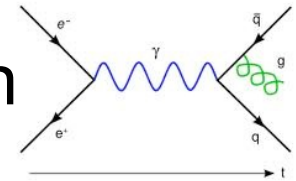
- Cheat by modifying action as:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2$$

giving the above e.o.m.

(Note wrong sign kinetic term for A^0)

- Quantise and impose gauge condition later
- This leads to 4 polarisations $\varepsilon^\lambda(\mathbf{p})$ that can be chosen such that: $\varepsilon^\kappa \cdot \varepsilon^\lambda = \eta^{\kappa\lambda}$
(1 timelike + 3 spacelike)



- For momentum $p \propto (1, 0, 0, 1)$ choose:

$$\varepsilon^0 = (1, 0, 0, 0)^T \quad \varepsilon^2 = (0, 0, 1, 0)^T$$

$$\varepsilon^1 = (0, 1, 0, 0)^T \quad \varepsilon^3 = (0, 0, 0, 1)^T$$

Physical polarisations are $\varepsilon^1, \varepsilon^2$. The others must somehow decouple.

- In fact, there is a serious problem with the timelike polarisation ε^0

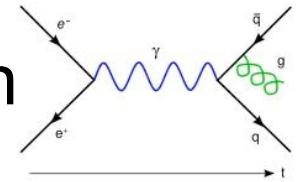
Commutation relations

$$[a^\kappa(\mathbf{p}), a^{\lambda\dagger}(\mathbf{q})] = -\eta^{\kappa\lambda} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

Notice - sign for $\kappa = \lambda = 0$, so the state

$a^{0\dagger}(\mathbf{q})|0\rangle$ has negative norm!

- At this point impose Lorenz gauge condition.



But how to impose Lorenz gauge? It appears problematic:

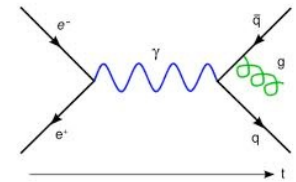
- Cannot impose as an operator equation $\partial_\mu A^\mu = 0$
(It's too strong: it violates the commutation relation for A^0, π^0)
- Cannot impose on physical states as $\partial_\mu A^\mu |\Psi\rangle = 0$
(Too strong: not even satisfied by the vacuum state!)
- Solution (Gupta-Bleuler) is to impose weaker condition on physical states:

$$\partial^\mu A_\mu^+ |\Psi\rangle = 0$$
 (+ sign refers to the +ve freq part of A_μ)

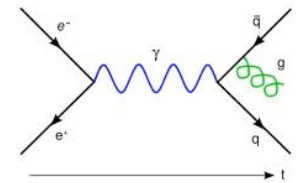
Implies $(a^3(\mathbf{p}) - a^0(\mathbf{p}))|\phi\rangle = 0$

- The unwanted timelike and longitudinal states combine into a null state of zero norm that decouples from the theory!

Take home message



- Developed picture in which particles arise naturally from perturbing quantum fields.
- QFT is not a theory, but a framework for constructing theories (that are local, causal and Lorentz invariant)
- However, it does have a handful of generic predictions:
 - There are 2 types of particles: bosons and fermions
 - All particles have their anti-particle
 - Couplings run with energy scale
(renormalisation, not covered in this course)
 - Forces are of the Yukawa/Coulomb type
(take non-relativistic limit of propagator and interpret as a potential in non-rel QM)
 - Symmetries can break or hide



In this course we have not covered:

- Quantization of vector field & QED (see QED course)
- Path Integral Formulation of QFT
- Renormalization
- Non-abelian Gauge Theories
- Non-perturbative effects (solitons, instantons)

Related courses: QED & the Standard Model
Higgs Boson Physics